

# A LOWER BOUND FOR $K_S^2$

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**ABSTRACT.** Let  $(S, \mathcal{L})$  be a smooth, irreducible, projective, complex surface, polarized by a very ample line bundle  $\mathcal{L}$  of degree  $d > 35$ . In this paper we prove that  $K_S^2 \geq -d(d-6)$ . The bound is sharp, and  $K_S^2 = -d(d-6)$  if and only if  $d$  is even, the linear system  $|H^0(S, \mathcal{L})|$  embeds  $S$  in a smooth rational normal scroll  $T \subset \mathbb{P}^5$  of dimension 3, and here, as a divisor,  $S$  is linearly equivalent to  $\frac{d}{2}Q$ , where  $Q$  is a quadric on  $T$ .

**Keywords:** Projective surface, Castelnuovo-Halphen's Theory, Rational normal scroll.

**MSC2010:** Primary 14J99; Secondary 14M20, 14N15, 51N35.

*Dedicated to Philippe Ellia on his sixtieth birthday.*

## 1. INTRODUCTION

The study of numerical invariants of projective varieties, and of the relations between them, is a classical subject in Algebraic Geometry. We refer to [16] and [17] for an overview on this argument. In this paper we turn our attention to the self-intersection of the canonical bundle of a smooth projective surface  $S$ . One already knows an upper bound in terms of the degree of  $S$  and of the dimension of the space where  $S$  is embedded [6]. Now we are going to prove the following *lower* bound:

**Theorem 1.1.** *Let  $(S, \mathcal{L})$  be a smooth, irreducible, projective, complex surface, polarized by a very ample line bundle  $\mathcal{L}$  of degree  $d > 35$ . Then:*

$$K_S^2 \geq -d(d-6).$$

*The bound is sharp, and the following properties are equivalent.*

- (i)  $K_S^2 = -d(d-6)$ ;
- (ii)  $h^0(S, \mathcal{L}) = 6$ , and the linear system  $|H^0(S, \mathcal{L})|$  embeds  $S$  in  $\mathbb{P}^5$  as a scroll with sectional genus  $g = \frac{d^2}{8} - \frac{3d}{4} + 1$ ;
- (iii)  $h^0(S, \mathcal{L}) = 6$ ,  $d$  is even, and the linear system  $|H^0(S, \mathcal{L})|$  embeds  $S$  in a smooth rational normal scroll  $T \subset \mathbb{P}^5$  of dimension 3, and here  $S$  is linearly equivalent to  $\frac{d}{2}(H_T - W)$ , where  $H_T$  is the hyperplane class of  $T$ , and  $W$  the ruling (i.e.  $S$  is linearly equivalent to an integer multiple of a smooth quadric  $Q \subset T$ ).

This is an inequality in the same vein of the classical Plücker-Clebsch formula

$$g \leq \frac{1}{2}(d-1)(d-2)$$

for the genus  $g$  of a projective curve of degree  $d$ . Unfortunately, the argument we developed does not enable us to state a sharp lower bound depending on the embedding dimension, like Castelnuovo's bound, neither to examine the case  $d \leq 35$ .

## 2. PROOF OF THEOREM 1.1

Put  $r + 1 := h^0(S, \mathcal{L})$ . Therefore  $|H^0(S, \mathcal{L})|$  embeds  $S$  in  $\mathbb{P}^r$ . Let  $H \subseteq \mathbb{P}^{r-1}$  be the general hyperplane section of  $S$ , so that  $\mathcal{L} \cong \mathcal{O}_S(H)$ . We denote by  $g$  the genus of  $H$ . If  $r = 2$  then  $d = 1$  and  $K_S^2 = 9 > 5$ . If  $r = 3$  then  $K_S^2 = d(d-4)^2 > -d(d-6)$  for  $d > 5$ . Therefore we may assume  $r \geq 4$ .

**The case  $r = 4$ .**

First we examine the case  $r = 4$ . In this case we only have to prove that, for  $d > 35$ , one has  $K_S^2 > -d(d-6)$ .

When  $r = 4$  we have the double point formula ([12], p. 433-434, Example 4.1.3):

$$(1) \quad d(d-5) - 10(g-1) + 12\chi(\mathcal{O}_S) - 2K_S^2 = 0$$

(use the adjunction formula  $2g-2 = H \cdot (H + K_S)$ ). Moreover by Lefschetz Hyperplane Theorem we know that the restriction map  $H^1(S, \mathcal{O}_S) \rightarrow H^1(H, \mathcal{O}_H)$  is injective. So, taking into account that

$$\chi(\mathcal{O}_S) = h^0(S, \mathcal{O}_S) - h^1(S, \mathcal{O}_S) + h^2(S, \mathcal{O}_S),$$

we get

$$(2) \quad \chi(\mathcal{O}_S) \geq 1 - g.$$

By (1) and (2) we deduce:

$$2K_S^2 \geq d(d-5) - 22(g-1).$$

Therefore to prove that  $K_S^2 > -d(d-6)$ , it is enough to prove that

$$(3) \quad 22(g-1) < 3d^2 - 17d.$$

First assume that  $S$  is not contained in a hypersurface of degree  $s < 5$ . In this case, since  $d > 14$ , then by Roth's Lemma [14], [13], we know that  $H$  is not contained in a surface of degree  $< 5$  in  $\mathbb{P}^3$ . Recall that the arithmetic genus of an irreducible, reduced, nondegenerate space curve of degree  $d > s^2 - s$ , not contained in a surface of degree  $< s$ , is bounded from above by the Halphen's bound [10]:

$$G(3; d, s) := \frac{d^2}{2s} + \frac{d}{2}(s-4) + 1 - \frac{(s-1-\epsilon)(\epsilon+1)(s-1)}{2s},$$

where  $\epsilon$  is defined by dividing  $d-1 = ms + \epsilon$ ,  $0 \leq \epsilon \leq s-1$ . Since  $d > 20$ , we may apply this bound with  $s = 5$ , and we have:

$$g \leq \frac{d^2}{10} + \frac{d}{2} + 1.$$

It follows (3) as soon as  $d > 35$ .

So we may assume that  $S$  is contained in an irreducible and reduced hypersurface of degree  $s \leq 4$ . First assume  $s \in \{2, 3\}$ . In this case one knows that for  $d > 12$

then  $S$  is of general type ([2], p. 213), therefore  $\chi(\mathcal{O}_S) \geq 1$  ([1], Théorème X.4, p. 154). Using this and (1), we see that a sufficient condition for  $K_S^2 > -d(d-6)$  is:

$$(4) \quad 10(g-1) < 3d^2 - 17d + 12.$$

If  $s = 2$  then by Halphen's bound we have  $g \leq \frac{d^2}{4} - d + 1$ . It follows (4) for  $d > 12$ . If  $s = 3$  then by Halphen's bound we have  $g \leq \frac{d^2}{6} - \frac{d}{2} + 1$ , from which it follows (4) for  $d > 7$ .

It remains to consider the case  $S$  is contained in an irreducible and reduced hypersurface of degree  $s = 4$ . In this case we need to refine previous analysis (in fact when  $s = 4$  one knows that  $S$  is of general type only for  $d > 97$  ([2], p. 213); moreover if one simply inserts Halphen's bound  $g \leq \frac{d^2}{8} + 1$  into (3), the inequality (3) is satisfied only for  $d > 68$ ). Now first recall that by ([8], Lemme 1) one has

$$\frac{d^2}{8} - \frac{9d}{8} + 1 \leq g \leq \frac{d^2}{8} + 1.$$

Hence there exists a rational number  $0 \leq x \leq 9$  such that

$$g = \frac{d^2}{8} + d \left( \frac{x-9}{8} \right) + 1.$$

If  $0 \leq x \leq 6$  then  $g \leq \frac{d^2}{8} - \frac{3d}{8} + 1$ , and (3) is satisfied for  $d > 35$ . So we may assume  $6 < x \leq 9$ . By ([5], Proposition 2, (2.2), (2.3) and proof) we have

$$\begin{aligned} \chi(\mathcal{O}_S) &\geq \frac{d^3}{96} - \frac{d^2}{16} - \frac{5d}{3} - \frac{333}{16} - (d-3)d \left( \frac{9-x}{8} \right) \\ &> \frac{d^3}{96} - \frac{d^2}{16} - \frac{5d}{3} - \frac{333}{16} - \frac{3d(d-3)}{8} = \frac{d^3}{96} - \frac{7d^2}{16} - \frac{13d}{24} - \frac{333}{16}. \end{aligned}$$

From (1) it follows that in order to prove that  $K_S^2 > -d(d-6)$ , it is enough that

$$10(g-1) \leq 3d^2 - 17d + 12 \left( \frac{d^3}{96} - \frac{7d^2}{16} - \frac{13d}{24} - \frac{333}{16} \right),$$

i.e. it is enough that

$$10(g-1) \leq \frac{d^3}{8} - \frac{9d^2}{4} - \frac{47d}{2} - \frac{999}{4}.$$

Taking into account that  $g \leq \frac{d^2}{8} + 1$ , one sees that previous inequality holds true for  $d > 35$ .

This concludes the proof of Theorem 1.1 in the case  $r = 4$ .

**The case  $r \geq 6$ .**

Now we are going to examine the case  $r \geq 6$ . Also in this case, we only have to prove that  $K_S^2 > -d(d-6)$ . We distinguish two cases, according that the line bundle  $\mathcal{O}_S(K_S + H)$  is spanned or not.

If  $\mathcal{O}_S(K_S + H)$  is spanned then  $(K_S + H)^2 \geq 0$ , therefore, taking into account the adjunction formula  $2g - 2 = H \cdot (H + K_S)$ , we get

$$K_S^2 \geq d - 4(g-1).$$

Let

$$G(r-1; d) = \frac{d^2}{2(r-2)} - \frac{rd}{2(r-2)} + \frac{(r-1-\epsilon)(1+\epsilon)}{2(r-2)}$$

be the Castelnuovo's bound for the genus of a nondegenerate integral curve of degree  $d$  in  $\mathbb{P}^{r-1}$ , that we may apply to  $g$  (here  $\epsilon$  is defined by dividing  $d-1 = m(r-2) + \epsilon$ ,  $0 \leq \epsilon \leq r-3$ ) ([7], Theorem (3.7), p. 87). So we deduce

$$\begin{aligned} K_S^2 + d(d-6) &\geq d - 4(G(r-1; d) - 1) + d(d-6) \\ &= \frac{1}{r-2} [(r-4)d^2 - (3r-10)d + 2(r + \epsilon^2 - \epsilon r + 2\epsilon - 3)]. \end{aligned}$$

Since  $r \geq 3$  and  $\epsilon \geq 0$ , we may write:

$$\begin{aligned} &(r-4)d^2 - (3r-10)d + 2(r + \epsilon^2 - \epsilon r + 2\epsilon - 3) \\ &\geq (r-4)d^2 - (3r-10)d - 2\epsilon r = d^2(r-4) - (5r-10)d + 2rd - 2\epsilon r. \end{aligned}$$

Observe that we have  $d \geq r-1$  for  $S$  is nondegenerate in  $\mathbb{P}^r$ . It follows  $2rd - 2\epsilon r > 0$  because  $\epsilon \leq r-3 < d$ . Hence, in order to prove that  $K_S^2 > -d(d-6)$  it suffices to prove that  $(r-4)d^2 - (5r-10)d \geq 0$ , i.e. that

$$d \geq \frac{5r-10}{r-4}.$$

Since  $d \geq r-1$ , this certainly holds for  $r \geq 9$ . On the other hand, an elementary direct computation shows that

$$(r-4)d^2 - (3r-10)d + 2(r + \epsilon^2 - \epsilon r + 2\epsilon - 3) > 0$$

holds true also for  $6 \leq r \leq 8$ ,  $0 \leq \epsilon \leq r-3$  and  $d \geq r-1$ , and for  $r=5$  and  $d > 5$ . Summing up, previous argument shows that

(5) if  $\mathcal{O}_S(K_S + H)$  is spanned,  $r \geq 5$  and  $d > 5$ , then  $K_S^2 > -d(d-6)$ .

Now we assume that  $\mathcal{O}_S(K_S + H)$  is not spanned. In this case one knows that  $S$  is a scroll ([15], Theorem (0.1)), i.e.  $S$  is a  $\mathbb{P}^1$ -bundle over a smooth curve  $C$ , and the restriction of  $\mathcal{O}_S(1)$  to a fibre is  $\mathcal{O}_{\mathbb{P}^1}(1)$  (either  $S$  is isomorphic to  $\mathbb{P}^2$ , but in this case  $K_S^2 = 9$ ). In particular one has that  $g$  is equal to the genus of  $C$ , and so we have ([12], Corollary 2.11, p. 374)

$$K_S^2 = 8(1-g).$$

Let

$$G(r; d) = \frac{d^2}{2(r-1)} - \frac{(r+1)d}{2(r-1)} + \frac{(r-\epsilon)(1+\epsilon)}{2(r-1)}$$

be the Castelnuovo's bound for the genus of a nondegenerate integral curve of degree  $d$  in  $\mathbb{P}^r$  (now  $\epsilon$  is defined by dividing  $d-1 = m(r-1) + \epsilon$ ,  $0 \leq \epsilon \leq r-2$ ) ([7], Theorem (3.7), p.87). Since  $g$  is equal to the genus of  $C$ , hence to the irregularity of  $S$ , by ([9], Lemma 4) we have:

$$g \leq G(r; d).$$

Hence we deduce:

$$K_S^2 = 8(1-g) \geq 8(1-G(r; d)),$$

and

$$K_S^2 + d(d-6) \geq 8(1-G(r; d)) + d(d-6) =: \psi(r; d),$$

with

$$\psi(r, d) = \left( \frac{r-5}{r-1} \right) (d^2 - 2d) - \frac{4}{r-1} (-r + 2 - \epsilon - \epsilon^2 + \epsilon r).$$

Taking into account that the function  $d \rightarrow d^2 - 2d$  is increasing for  $d \geq 1$ , and that  $d \geq r - 1$ , we have:

$$\psi(r, d) \geq \psi(r, r - 1) = \frac{1}{r - 1} (r^3 - 9r^2 + 27r - 23 + 4\epsilon + 4\epsilon^2 - 4\epsilon r).$$

Now we notice:

$$\begin{aligned} r^3 - 9r^2 + 27r - 23 + 4\epsilon + 4\epsilon^2 - 4\epsilon r &\geq r^3 - 9r^2 + 27r - 23 + 4\epsilon^2 - 4\epsilon r \\ &= r^3 - 10r^2 + 27r - 23 + (r - 2\epsilon)^2 \geq r^3 - 10r^2 + 27r - 23, \end{aligned}$$

which is  $> 0$  for  $r \geq 7$ . An elementary direct computation proves that  $\psi(r, d) > 0$  also for  $r = 6$  (and  $d > 4$ ). This concludes the proof of Theorem 1.1 in the case  $r \geq 6$ .

*Remark 2.1.* We also remark that for  $r = 5$  we have  $\psi(5, d) = \epsilon^2 - 4\epsilon + 3$ . Since

$$\epsilon^2 - 4\epsilon + 3 = \begin{cases} 3 & \text{if } \epsilon = 0 \\ 0 & \text{if } \epsilon \in \{1, 3\} \\ -1 & \text{if } \epsilon = 2, \end{cases}$$

taking into account (5), it follows that  $K_S^2 > -d(d - 6)$  holds true also for  $r = 5$  and  $d > 5$ , unless  $S \subset \mathbb{P}^5$  is a scroll,  $K_S^2 = 8(1 - g)$ , and

$$(6) \quad g = G(5; d) = \frac{1}{8}d^2 - \frac{3}{4}d + \frac{(5 - \epsilon)(\epsilon + 1)}{8},$$

with  $d - 1 = 4m + \epsilon$ ,  $0 < \epsilon \leq 3$ . We will use this fact in the analysis of the case  $r = 5$  below.

### The last case: $r = 5$ .

In this section we examine the case  $r = 5$ ,  $S \subset \mathbb{P}^5$ .

By previous remark, we know that for  $d > 5$  one has  $K_S^2 > -d(d - 6)$ , except when the surface  $S$  satisfies the condition  $g = G(5; d)$ . Now we are going to prove that these exceptions are necessarily contained in a smooth rational normal scroll of dimension 3. As an intermediate step we prove that such surfaces are contained in a threefold of degree  $\leq 4$  (when  $d > 30$ ).

To this purpose, assume that  $S$  is as before, and that it is not contained in a threefold of degree  $< 5$ . By ([3], Theorem (0.2)) we know that if  $d > 24$  then  $H$  is not contained in a surface of degree  $< 5$  in  $\mathbb{P}^4$ . Then by ([7], Theorem (3.22), p. 117) we deduce that for  $d > 143$  one has

$$g \leq G(4; d, 5) := \frac{1}{10}d^2 - \frac{3}{10}d + \frac{1}{5} + \frac{1}{10}v - \frac{1}{10}v^2 + w,$$

where  $v$  is defined by dividing  $d - 1 = 5n + v$ ,  $0 \leq v \leq 4$ , and  $w := \max\{0, [\frac{v}{2}]\}$  (with the notation of [7] we have  $\pi_2(d, 4) = G(4; d, 5)$ ). An elementary computation proves that

$$(7) \quad G(4; d, 5) - G(5; d) < 0$$

for  $d > 18$ . This is absurd, therefore if  $K_S^2 \leq -d(d-6)$  and  $d > 143$ , then  $S$  is contained in a threefold of degree  $\leq 4$ . In order to prove this also for  $30 < d < 144$  we have to refine previous analysis. To this aim, first recall that

$$G(4; d, 5) = \sum_{i=1}^{+\infty} (d - h(i)),$$

where

$$h(i) := \begin{cases} 5i - 1 & \text{if } 1 \leq i \leq n \\ d - w & \text{if } i = n + 1 \\ d & \text{if } i \geq n + 2 \end{cases}$$

([7], p. 119). Let  $\Gamma \subset \mathbb{P}^3$  be the general hyperplane section of  $H$ , and let  $h_\Gamma$  be its Hilbert function.

Assume first that  $h^0(\mathbb{P}^3, \mathcal{I}_\Gamma(2)) \geq 2$ . Then, if  $d > 4$ , by monodromy ([4], Proposition 2.1),  $\Gamma$  is contained in a reduced and irreducible space curve of degree  $\leq 4$ . By ([3], Theorem (0.2)) we deduce that, for  $d > 20$ ,  $S$  is contained in a threefold of degree  $\leq 4$ . Hence we may assume  $h^0(\mathbb{P}^3, \mathcal{I}_\Gamma(2)) \leq 1$ .

Assume now  $h^0(\mathbb{P}^3, \mathcal{I}_\Gamma(2)) = 1$ , and  $h^0(\mathbb{P}^3, \mathcal{I}_\Gamma(3)) > 4$ . As before, if  $d > 6$ , by monodromy ([4], Proposition 2.1),  $\Gamma$  is contained in a reduced and irreducible space curve  $X$  of degree  $\deg(X) \leq 6$ . Again as before, if  $\deg(X) \leq 4$ , then  $S$  is contained in a threefold of degree  $\leq 4$ . So we may assume  $5 \leq \deg(X) \leq 6$ . By ([4], Proposition 4.1) we know that, when  $d > 30$ ,

$$h_\Gamma(i) \geq h(i) \quad \text{for any } i \geq 0.$$

Hence we have ([7], Corollary (3.2), p. 84):

$$g \leq \sum_{i=1}^{+\infty} (d - h_\Gamma(i)) \leq \sum_{i=1}^{+\infty} (d - h(i)) = G(4; d, 5).$$

Since  $g = G(5; d)$ , this is absurd for  $d > 18$  (compare with (7)). If  $h^0(\mathbb{P}^3, \mathcal{I}_\Gamma(2)) = 1$  and  $h^0(\mathbb{P}^3, \mathcal{I}_\Gamma(3)) = 4$ , then we have

$$h_\Gamma(1) = 4, \quad h_\Gamma(2) = 9, \quad h_\Gamma(3) = 16.$$

Using induction and ([7], Corollary (3.5), p. 86) we get for any  $i \geq 4$ :

$$(8) \quad h_\Gamma(i) \geq \min\{d, h_\Gamma(i-3) + h_\Gamma(3) - 1\} \geq \min\{d, h(i-3) + 15\} \geq h(i).$$

As before, this leads to  $g \leq G(4; d, 5)$ , which is absurd for  $d > 18$ .

Next assume  $h^0(\mathbb{P}^3, \mathcal{I}_\Gamma(2)) = 0$ , and that  $h^0(\mathbb{P}^3, \mathcal{I}_\Gamma(3)) \leq 1$ . Then we have:  $h_\Gamma(1) = 4$ ,  $h_\Gamma(2) = 10$ ,  $h_\Gamma(3) \geq 19$ . Then a similar computation as before leads to a contradiction if  $d > 18$ .

Finally assume  $h^0(\mathbb{P}^3, \mathcal{I}_\Gamma(2)) = 0$ , and  $h^0(\mathbb{P}^3, \mathcal{I}_\Gamma(3)) \geq 2$ . Then, by monodromy ([4], Proposition 2.1),  $\Gamma$  is contained in a reduced and irreducible curve  $X \subset \mathbb{P}^3$  of degree  $\deg(X) \leq 9$ . By ([4], Proposition 4.1) we may also assume  $\deg(X) \geq 7$ . Let  $X' \subset \mathbb{P}^2$  be the general hyperplane section of  $X$ . By Castelnuovo's Theory ([7],

Lemma (3.1), p. 83) we know that:

$$h_X(i) \geq \sum_{j=0}^i h_{X'}(j).$$

Therefore, taking into account ([7], Corollary (3.6), p. 87), we have  $h_X(1) \geq 4$ ,  $h_X(2) \geq 9$ ,  $h_X(3) \geq 16$ . On the other hand, since  $d > 27$ , by Bezout's Theorem we have  $h_\Gamma(i) = h_X(i)$  for any  $1 \leq i \leq 3$ . Hence we may repeat the same argument as in (8), obtaining  $g \leq G(4; d, 5)$ , which is absurd.

Summing up, we proved that if  $r = 5$ ,  $d > 30$  and  $K_S^2 \leq -d(d-6)$ , then  $S$  is a scroll,  $K_S^2 = 8(1-g)$ ,  $g = G(5; d)$ ,  $d \not\equiv 1 \pmod{4}$ , and  $S$  is contained in a threefold  $T \subset \mathbb{P}^5$  of degree  $\leq 4$ . Unfortunately, assuming  $S$  is not contained in a threefold of degree  $< 4$ , previous argument does not work. Therefore we need a different argument to prove that  $S$  cannot lie in a threefold of degree 4.

To this aim, assume by contradiction that  $S$  is contained in a threefold of degree 4. Recall that we are assuming that  $S$  is a scroll,  $K_S^2 = 8(1-g)$ ,  $g = G(5; d)$ ,  $d \not\equiv 1 \pmod{4}$ , and that  $d > 30$ . In particular we have (compare with (6)):

$$(9) \quad g \geq \frac{1}{8}d^2 - \frac{3}{4}d + 1.$$

On the other hand, by ([7], p. 98-99) we know that

$$h_\Gamma(i) \geq k(i) := \begin{cases} 4i & \text{if } 1 \leq i \leq p \\ d-1 & \text{if } i = p+1 \text{ and } q = 3 \\ d & \text{if } i = p+1 \text{ and } q < 3 \text{ or } i \geq p+2, \end{cases}$$

where  $p$  is defined by dividing  $d-1 = 4p+q$ ,  $0 \leq q \leq 3$ . It follows that

$$g \leq \sum_{i=1}^{+\infty} (d - h_\Gamma(i)) \leq \sum_{i=1}^{+\infty} (d - k(i)) = G(4; d, 4),$$

with

$$G(4; d, 4) = \frac{1}{8}d^2 - \frac{1}{2}d + \frac{3}{8} + \frac{1}{4}q - \frac{1}{8}q^2 + t,$$

where  $t = 0$  if  $0 \leq q \leq 2$ , and  $t = 1$  if  $q = 3$  (with the notation as in ([7], p. 99) we have  $G(4; d, 4) = \pi_1(d, 4)$ ). Moreover, since  $S$  is a scroll, we also have

$$\chi(\mathcal{O}_S) = 1 - g.$$

And using the same argument as in the proof of ([5], Proposition 1, (1.2)), we get:

$$\begin{aligned} \chi(\mathcal{O}_S) = 1 - g &\geq 1 + \sum_{i=1}^{d-4} (i-1)(d-k(i)) - (d-4) \left( \sum_{i=1}^{d-4} (d-k(i)) - g \right) \\ &= 1 + \sum_{i=1}^{d-4} (i-1)(d-k(i)) - (d-4)(G(4; d, 4) - g). \end{aligned}$$

Hence we have

$$(d-3)g \leq - \sum_{i=1}^{d-4} (i-1)(d-k(i)) + (d-4)G(4; d, 4).$$

Using (9) we get:

$$(d-3) \left( \frac{1}{8}d^2 - \frac{3}{4}d + 1 \right) \leq - \sum_{i=1}^{d-4} (i-1)(d-k(i)) + (d-4)G(4; d, 4).$$

Taking into account that

$$\sum_{i=1}^{d-4} (i-1)(d-k(i)) = \binom{p}{2}d - 8 \binom{p+1}{3} + tp,$$

previous inequality is equivalent to:

$$-d^3 + 24d^2 + (-9q^2 + 18q - 125 + 72t)d - 2q^3 + 42q^2 - 70q + 174 - 360t + 24tq \geq 0.$$

This is impossible if  $d > 24$  (recall that  $d-1 = 4p+q$ ,  $0 \leq q \leq 3$ , and that  $t = 0$  for  $0 \leq q \leq 2$ , and that  $t = 1$  for  $q = 3$ ).

So we proved that if  $d > 30$  and  $K_S^2 \leq -d(d-6)$ , then  $S$  is a scroll,  $g = G(5; d)$ , and it is contained in a threefold  $T \subset \mathbb{P}^5$  of minimal degree 3, i.e. in a rational normal scroll  $T \subset \mathbb{P}^5$  of dimension 3 and degree 3 ([11], p. 51).

First we prove that  $T$  is necessarily nonsingular. Suppose not. Let  $L$  be a general hyperplane passing through a singular point of  $T$ . Then  $H \subset L$  is a curve contained in the surface  $T' := T \cap L$ , which is a singular rational normal scroll. Put  $d-1 = 3p+q$ ,  $0 \leq q \leq 2$ . Since the divisor class group of  $T'$  is generated by a line of the ruling, then  $H$  is residual to  $2-q$  lines of the ruling of  $T'$ , in a complete intersection of  $T'$  with a hypersurface of degree  $p+1$ . Therefore  $H$  is a.C.M., and so also  $S$  is. In particular the arithmetic genus of  $S$  is equal to the geometric genus, therefore  $\chi(\mathcal{O}_S) = 1 - g \geq 1$ , i.e.  $g = 0$ , which is impossible in view of the inequality  $g \geq \frac{1}{8}d^2 - \frac{3}{4}d + 1$ .

To conclude the proof of the Theorem it suffices to prove the following:

**Proposition 2.2.** *Let  $S \subset \mathbb{P}^5$  be a nondegenerate, smooth, irreducible, projective, complex surface of degree  $d \geq 18$ , contained in a smooth rational normal scroll  $T$  of dimension 3. Then  $K_S^2 \geq -d(d-6)$ . The bound is sharp, and the following properties are equivalent.*

- (i)  $K_S^2 = -d(d-6)$ ;
- (ii)  $S$  is a scroll with sectional genus  $g = \frac{d^2}{8} - \frac{3d}{4} + 1$ ;
- (iii)  $S$  is linearly equivalent to  $\frac{d}{2}(H_T - W)$ , where  $H_T$  is the hyperplane class of  $T$ , and  $W$  the ruling.

Before proving this, we need the following lemma:

**Lemma 2.3.** *Let  $T \subset \mathbb{P}^5$  be a nonsingular rational normal scroll of dimension 3. Let  $H_T$  be a hyperplane section of  $T$ , and  $W$  a plane of the ruling. Let  $\alpha$  and  $\beta$  be integer numbers. Then the linear system  $|\alpha H_T + \beta W|$  contains an irreducible, nonsingular, and nondegenerate surface if and only if  $\alpha > 0$ ,  $\alpha + \beta \geq 0$ , and  $3\alpha + \beta \geq 4$ .*



*Proof of Lemma 2.3.* First assume that  $|\alpha H_T + \beta W|$  contains an irreducible, non-singular, and nondegenerate surface  $S$ . Let  $T' := T \cap \mathbb{P}^4$  be a general hyperplane section of  $T$ , which is a rational normal scroll surface in  $\mathbb{P}^4$ . Let  $H_{T'}$  be a hyperplane section of  $T'$ , and  $W'$  a line of the ruling of  $T'$ . Using the same notation of ([12], Notation 2.8.1, p. 373, Example 2.19.1, p. 381), we have  $C_0 = H_{T'} - 2W'$ ,  $C_0^2 = -e = -1$ . Therefore the general hyperplane section of  $S$  belongs to the linear system  $|\alpha H_{T'} + \beta W'| = |\alpha C_0 + (2\alpha + \beta)W'|$ . Taking into account that  $S$  is nondegenerate, then by ([12], Corollary 2.18, p. 380) we get  $\alpha > 0$ ,  $\alpha + \beta \geq 0$ , and  $\deg(S) = 3\alpha + \beta \geq 4$ .

Conversely, assume  $\alpha > 0$  and  $\alpha + \beta \geq 0$ . Using the same argument as in the proof of ([6], Proposition 2.3), we see that the linear system  $|\alpha H_T + \beta W|$  is non empty, and base point free. By Bertini's Theorem it follows that its general member is nonsingular. As for the irreducibility, consider the exact sequence:

$$\begin{aligned} 0 \rightarrow \mathcal{O}_T((\alpha - 1)H_T + \beta W) \rightarrow \mathcal{O}_T(\alpha H_T + \beta W) \rightarrow \\ \rightarrow \mathcal{O}_T(\alpha H_T + \beta W) \otimes \mathcal{O}_{T'} = \mathcal{O}_{T'}(\alpha H_{T'} + \beta W') \rightarrow 0. \end{aligned}$$

Since  $K_T \sim -3H_T + W$  then we may write:

$$(\alpha - 1)H_T + \beta W = K_T + (\alpha + 2)H_T + (\beta - 1)W.$$

As before, by [6], we know that the line bundle  $\mathcal{O}_T((\alpha + 2)H_T + (\beta - 1)W)$  is spanned, hence nef. On the other hand we have

$$((\alpha + 2)H_T + (\beta - 1)W)^3 = 3(\alpha + 2)^2(\alpha + \beta + 1) > 0.$$

Therefore  $\mathcal{O}_T((\alpha + 2)H_T + (\beta - 1)W)$  is big and nef. Then by Kawamata-Viehweg Theorem we deduce

$$H^1(\mathcal{O}_T((\alpha - 1)H_T + \beta W)) = 0.$$

This implies that the linear system  $|\alpha H_T + \beta W|$  cut on  $T'$  the complete linear system  $|\mathcal{O}_{T'}(\alpha H_{T'} + \beta W')|$ , whose general member is irreducible by ([12], Corollary 2.18, p. 380). A fortiori the general member of  $|\alpha H_T + \beta W|$  is irreducible.

Finally we notice that the general  $S \in |\alpha H_T + \beta W|$  is nondegenerate. In fact otherwise we would have  $S = H_T$ , which is in contrast with our assumption  $\deg(\alpha H_T + \beta W) = 3\alpha + \beta \geq 4$ .  $\square$

*Proof of Proposition 2.2.* Define  $m$  and  $\epsilon$  by diving

$$(10) \quad d - 1 = 3m + \epsilon, \quad 0 \leq \epsilon \leq 3.$$

Since the Picard group of  $T$  is freely generated by the hyperplane class  $H_T$  of  $T$  and by the plane  $W$  of the ruling, then there exists an unique integer  $a \in \mathbb{Z}$  such that

$$S \sim (m + 1 + a)H_T + (\epsilon + 1 - 3(a + 1))W.$$

By previous lemma, we may restrict our analysis to the range

$$-m \leq a \leq \frac{1}{2}(m + \epsilon - 1).$$

Taking into account that

$$K_T \sim -3H_T + W,$$

from the adjunction formula we get (compare with [6], (0.4) and p. 149)

$$\begin{aligned} K_S^2 = \phi(a) &= -6a^3 + a^2(-9m + 5 + 3\epsilon) + a(2m(3\epsilon - 4) - 6\epsilon + 10) \\ &\quad + 3m^3 + m^2(3\epsilon - 13) + m(10 - 6\epsilon) + 8. \end{aligned}$$

In the given range this function takes its minimum exactly when

$$a = a^* := \frac{1}{2}(m + \epsilon - 1)$$

(see Appendix below). Since  $\phi(a^*) = -d(d - 6)$ , it follows  $K_S^2 > -d(d - 6)$ , except when  $d$  is even and

$$\frac{d}{2} = m + 1 + a^* = -(\epsilon + 1 - 3(a^* + 1)).$$

In this case we already know that  $S$  is a scroll with  $g = \frac{d^2}{8} - \frac{3d}{4} + 1$ .  $\square$

### Appendix.

With the notation as in the proof of Proposition 2.2, consider the function

$$\begin{aligned} \phi(a) &:= -6a^3 + a^2(-9m + 5 + 3\epsilon) + a(2m(3\epsilon - 4) - 6\epsilon + 10) \\ &\quad + 3m^3 + m^2(3\epsilon - 13) + m(10 - 6\epsilon) + 8. \end{aligned}$$

We are going to prove that *if  $d \geq 18$  and  $-m \leq a \leq \frac{1}{2}(m + \epsilon - 1)$ , then  $\phi(a) \geq -d(d - 6)$ , and  $\phi(a) = -d(d - 6)$  if and only if  $a = a^*$* . To this purpose we derive with respect to  $a$ :

$$\phi'(a) = -18a^2 + 2a(-9m + 5 + 3\epsilon) + 2m(3\epsilon - 4) - 6\epsilon + 10.$$

This is a degree 2 polynomial in the variable  $a$ , whose discriminant is:

$$\Delta = 324m^2 - 936m + 216m\epsilon + 110 + 114\epsilon + 36\epsilon^2,$$

which is  $> 0$  when  $m \geq 3$ , hence when  $d \geq 12$  (compare with (10)). Denote by  $a_1$  and  $a_2$  the real roots of the equation  $\phi'(a) = 0$ , with  $a_1 < a_2$ , and let  $I$  be the open interval  $I = (a_1, a_2)$ . Then  $\phi'(a) > 0$  if and only if  $a \in I$ . In particular  $\phi(a)$  is strictly increasing for  $a \in I$ , and strictly decreasing for  $a \notin I$ . Now observe that

$$\phi(-m) = 8, \quad \phi(-m + 1) = -9m + 17 - 3\epsilon, \quad \phi(-m + 2) = 0.$$

Notice that  $-9m + 17 - 3\epsilon \geq -9m + 11$  because  $0 \leq \epsilon \leq 2$ ,  $-9m + 11 \geq -9\frac{d-1}{3} + 11$  since  $\frac{d-1}{3} \leq m \leq \frac{d-3}{3}$ , and  $-3(d - 1) + 11 > -d(d - 6)$  if  $d > 7$ . So for  $a \in \{-m, -m + 1, -m + 2\}$  we have  $K_S^2 > -d(d - 6)$ . Moreover we have  $\phi'(-m + 2) = 18m + 6\epsilon - 42 > 0$  if  $d \geq 12$ , and  $\phi'(-1) = 10m + 6m\epsilon - 12\epsilon - 18 > 0$  if  $d \geq 9$ . Therefore  $[-m + 2, -1] \subset I$  and so  $\phi(a) \geq \phi(-m + 2) = 0 > -d(d - 6)$  for  $-m \leq a \leq -1$  and  $d > 12$ . We also have:

$$\phi(0) = (m - 2)(3m^2 - 7m + 3m\epsilon - 4),$$

which is  $\geq 0$  for  $m \geq 3$ , hence for  $d \geq 12$ . And

$$\phi(1) = (m - 1)(3m^2 - 10m + 3m\epsilon + 3\epsilon - 17),$$

which is  $\geq 0$  for  $m \geq 5$ , hence for  $d \geq 18$ . Moreover we have:

$$\phi'(1) = 2 - 26m + 6m\epsilon < 0.$$

Therefore  $\phi(a)$  is strictly decreasing for  $a \geq 1$ . It follows that in the range  $1 \leq a \leq a^* := \lfloor \frac{m+\epsilon-1}{2} \rfloor$ , the function  $\phi$  takes its minimum exactly when  $a = a^*$ . Define  $p$  and  $q$  by dividing:

$$m + \epsilon - 1 = 2p + q, \quad 0 \leq q \leq 1,$$

so that  $p = a^*$ . Notice that  $d$  is even if and only if  $q = 0$ . We have:

$$\phi(a^*) = \begin{cases} -d(d-6) & \text{if } q = 0 \\ -\frac{1}{4}d^2 + \frac{1}{2}d + \frac{35}{4} & \text{if } q = 1. \end{cases}$$

Since when  $d > 5$  we have

$$-\frac{1}{4}d^2 + \frac{1}{2}d + \frac{35}{4} > -d(d-6),$$

by previous analysis it follows that, for any integer  $-m \leq a \leq \frac{m+\epsilon-1}{2}$ , one has  $\phi(a) \geq -d(d-6)$ , and  $\phi(a) = -d(d-6)$  if and only if  $d$  is even and  $a = \frac{m+\epsilon-1}{2}$ .

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